299. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

The sides of a triangle and the area are in arithmetical progression. Find their values, and show there is only *one* solution in rational integers.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

To avoid fractions, take the sides and area to be, in order, $2x$, $2x+2y$, $2x+4y$, and $2x+6y$; then $3(x+y)$=the half sum of the sides, from which we have

$$[3(x+y)(x+3y)(x+y)(x-y)]^{\frac{1}{2}}=2x+6y\ldots(1).$$

Throw off the radical and divide each side of equation (1) by $(x+3y)(x+y)^2$ and we have after reduction

$$3(x-y)/(x+3y)=4/(x+y)^2=m^2\ldots(2).$$

The least value of m for positive integral results=1. Therefore $x=2-y$, and the sides and area in order are, $4-2y$, 4, $4+2y$, and $4+4y$.

The least value of y for positive integral results$=\frac{1}{2}$. Therefore, the sides and area are in order 3, 4, 5, and 6.

The triangle is a right triangle; and there are an indefinite number of similar triangles; integral or fractional multiples; but there is but *one* solution.

Also solved by George W. Hartwell, T. I. Wodo, G. B. M. Zerr, M. V. Spunar, A. H. Holmes, J. Scheffer, and J. M. Arnold.

172. Proposed by DR. L. E. DICKSON, The University of Chicago.

Without solving the algebraically solvable quintic, $y^5+py^3+\frac{1}{5}p^2y+r=0$, prove that it is irreducible in the domain of rationality (p, r).

Solution by H. S. VANDIVER, Bala, Pa.

Put the function in the form

$$5y^5+5py^3+p^2y+5r.$$

If the original function is reducible in domain (p, r) this function is also. The assumption that it is reducible in domain (p, r) is equivalent to the assumption that it can be expressed as the product of two factors:

$$x^n+\alpha_1x^{n-1}+\ldots+\alpha_n,$$
$$5x^{5-n}+\beta_1x^{4-n}+\ldots+\beta_{5-n},$$

where the α's and β's are rational functions in p and r. By Theorem VI, page 79, Vol. I, of Weber's *Algebra*, French edition, they may also be considered integral. The form of the factors shows that the function remains

reducible for all finite values of p and r. Let $p=5$, $r=5$. Then, ignoring constant factor 5, y^5+5y^3+5y+5 is reducible. But this is irreducible by the well known theorem of Eisenstein (Weber, l. c., p. 702).

The irreducibility may also be proved by setting $p=0$, $r=2$, whence the function y^5+2, which is irreducible by the theorem in Dickson's *Theory of Algebraic Equations*, p. 77, §90.

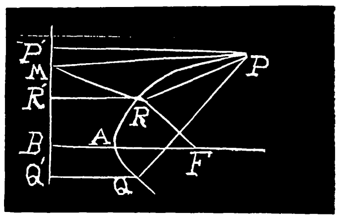
GEOMETRY.

332. Proposed by DAVID F. KELLEY, New York, N. Y.

To find the area of a parabolic sector, by a hitherto unpublished method.

Solution by the PROPOSER.

Let a secant meet a parabola in the points P and Q. Join the points P and Q to F which is the focus of the parabola. Let fall perpendiculars from P and Q on the directrix, BC, meeting it in the points P' and Q', respectively. Let R be any other point on the curve between P and Q. Join P and R, and let fall RR' perpendicular to directrix BC, and meeting it in the point R'. Bisect $P'R'$ in M, and join M and R, and M and P. By a well known geometrical theorem, area of $\triangle PMR=\frac{1}{2}$ quadrilateral $PP'R'R$. Let R' move indefinitely near to P', then, in the limit, $MR=R'R=FR$, and $NP=P'P=FP$. Therefore, in the limit, $\triangle PRF=\triangle PMR=\frac{1}{2}$ quadrilateral $PP'R'R$. Hence, it readily follows that space $FPRAQ=\frac{1}{2}$ space $PRAQQ'P'$. Hence, if $O=$space $PRAQQ'P'$, and $I=$space $PRAQ$, and $\triangle'=$area $\triangle FPQ$, and $k=$area of quadrilateral $PQQ'P'$, we have the following two equations connecting O and I:



$$\triangle'+I=\tfrac{1}{2}O,\quad O+I=k.$$

In particular, when PQ is perpendicular to AB, if x and y be coordinates of P, we have $(a-x)y+I=\frac{1}{2}O$. $I+O=2(a+x)y$, and solving for I we get $I=4xy/3$.

Again, since, in the limit, $\triangle FPR=\triangle MPR$, it follows that if perpendiculars be let fall from P' and F on the tangent to the parabola at P, then these perpendiculars are equal, and hence it is readily seen since $FP=PP'$ that the line joining F and P' is bisected by the tangent at P, and is at right angles to it.

Also solved by G. B. M. Zerr, and H. V. Spunar.